Math 1B, Fall 2008

Recursive Sequence Example

Define the sequence $\{a_n\}$ by

$$\begin{aligned} a_1 &= 1 \\ a_{n+1} &= \sqrt[3]{a_n + 6}. \end{aligned}$$

We will prove that $\{a_n\}$ converges, and find the limit. This kind of sequence, where $a_{n+1}$ is defined in terms of $a_n$, is called *recursively defined.* We've done a couple of problems where we could only find the limit of a recursively defined sequence after we already knew that the sequence converged (for example, see problem 3 on Section 108 Quiz 4 Solutions). So here is the outline we will follow:

(1) Prove that $\{a_n\}$ converges, via (a),(b) and (c):

- (a) Prove that $\{a_n\}$ is bounded.
- (b) Prove that $\{a_n\}$ is monotonic.
- (c) Use the Monotonic Sequence Theorem.

(2) Find $\lim_{n\to\infty} a_n$.

Start with **(1a)**. We will prove $\{a_n\}$ is bounded, by induction. In particular, we will show that $0 \le a_n \le 2$ for all $n \ge 1$.

[In a problem, you will most likely be given these bounds (0 and 2). If you are not, then you'll have to try to guess them.]

Here is the proof that $0 \le a_n \le 2$ for $n \ge 1$. First, the base case $\underline{n = 1}$. Since $a_1 = 1$, we have $0 \le a_1 \le 2$. Next, the inductive step $\underline{k \implies k+1}$. So assume $0 \le a_k \le 2$ and we will prove $0 \le a_{k+1} \le 2$. Since $a_k$ is positive, $a_{k+1} = \sqrt[3]{a_n + 6}$ must also be positive. So $a_{k+1} \ge 0$. Also,

$$\begin{aligned} a_{k+1} &= \sqrt[3]{a_k + 6} \\ &\le \sqrt[3]{2 + 6} \qquad \text{(because } a_k \le 2\text{, and } \sqrt[3]{\phantom{x}} \text{ is increasing)} \\ &= 2. \end{aligned}$$

So $a_{k+1} \le 2$. We have shown $0 \le a_{k+1} \le 2$, which finishes the inductive proof that $0 \le a_n \le 2$ for all $n \ge 1$.

Now we do **(1b)** and show that $a_n$ is monotonic. First, some general thoughts on this part of the problem. Define $f(x) = \sqrt[3]{x + 6}$, so $f(a_n) = a_{n+1}$. As a rule, if $f$ is increasing on the interval of possible values of $\{a_n\}$ ($0 \leq x \leq 2$ in this case), then the sequence $\{a_n\}$ is monotonic. It is tempting to think that $f$ increasing means that $\{a_n\}$ is increasing, but that is not the case! If $f$ is increasing, then to see if the sequence is increasing or decreasing, just see if you have $a_1 < a_2$ or $a_1 > a_2$.

In this case, the sequence should be increasing. We now prove that $a_n < a_{n+1}$ for all $n \geq 1$, by induction. First, the base case $\underline{n = 1}$. We can find $a_2 = \sqrt[3]{7} > 1 = a_1$, so $a_1 < a_2$. Now the inductive step $\underline{k \implies k+1}$. So we assume that $a_k < a_{k+1}$, and prove that $a_{k+1} < a_{k+2}$. By our definition of $f$ above, $f(a_k) = a_{k+1}$ and $f(a_{k+1}) = a_{k+2}$. We now check that $f$ is increasing for $0 \leq x \leq 2$, the possible values of $a_n$ (by step 1a). To do this, differentiate to get

$$\begin{aligned} f'(x) &= \frac{1}{3}(x+6)^{-2/3} \\ &\geq 0 \qquad \text{for } 0 \leq x \leq 2. \end{aligned}$$

Since $f$ is increasing, and $a_k < a_{k+1}$, we get $f(a_k) < f(a_{k+1})$ (the idea is that if $f$ is increasing, then "bigger inputs give bigger outputs"). But this means $a_{k+1} < a_{k+2}$, which finishes the proof by induction that $\{a_n\}$ is increasing.

We have shown that $\{a_n\}$ is bounded and increasing, so by the Monotone Sequence Theorem $\{a_n\}$ converges. This was **(1c)**.

We now turn to step **(2)**, which is finding the limit of $\{a_n\}$. Let $L = \lim_{n\to\infty} a_n$. Then we also have $L = \lim_{n\to\infty} a_{n+1}$ (essentially, all we have done is throw out the first element of the sequence). So starting with

$$a_{n+1} = \sqrt[3]{a_n + 6}$$

and taking limits of both sides gives

$$\begin{aligned} L &= \lim_{n\to\infty} a_{n+1} \\ &= \lim_{n\to\infty} \sqrt[3]{a_n + 6} \\ &= \sqrt[3]{\lim_{n\to\infty} a_n + 6} \qquad \text{(limit rules/continuity)} \\ &= \sqrt[3]{L+6}. \end{aligned}$$

So $L = \sqrt[3]{L+6}$. Cubing both sides, rearranging, and then factoring gives

$$0 = L^3 - L - 6 = (L-2)(L^2 + 2L + 3).$$

So $L$ is root of the polynomial $(x-2)(x^2+2x+3)$. Since $x^2 + 2x + 3$ has no real roots, we must have $L = 2$. In conclusion,

$$\lim_{n\to\infty} a_n = 2.$$

So this is a pretty long example, and you wouldn't have to write so much if you did it. But here's a checklist of important steps

- Prove by induction that $a_n$ is bounded, $p \leq a_n \leq q$ for all $n \geq 1$. You may have to use some algebra to get this to work, depending on the problem.
- Define $f(x)$ so that $f(a_n) = a_{n+1}$.
- Make sure $f(x)$ is increasing (i.e. $f'(x) \geq 0$) for $p \leq x \leq q$.
- Prove that $\{a_n\}$ is increasing (or decreasing), by induction. To choose between increasing or decreasing, just check whether $a_1 < a_2$ or vice versa. This proof will always be pretty much the same as the one in the example above.
- Conclude that $\{a_n\}$ converges, and let $L = \lim_{n\to\infty} a_n$.
- Set $f(L) = L$, and solve for $L$. If there is more than one such $L$, use your knowledge of the sequence to eliminate all but one of them. For example, if $a_n$ is always positive, you can eliminate negative choices. Similarly, if $a_1 = 1$ and $a_n$ is increasing, the limit cannot be $1/2$.